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Positive topological entropy of Reeb flows on spherizations

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Abstract

Let M be a closed manifold whose based loop space $\Omega(M)$ is "complicated". Examples are rationally hyperbolic manifolds and manifolds whose fundamental group has exponential growth. Consider a hypersurface Σ in T^*M which is fiberwise starshaped with respect to the origin. Choose a function $H : T^*M \to \mathbb{R}$ such that Σ is a regular energy surface of H, and let φ^t be the restriction to Σ of the Hamiltonian flow of H.

THEOREM 1. *The topological entropy of* φ^t *is positive.*

This result has been known for fiberwise *convex* Σ by work of Dinaburg, Gromov, Paternain, and Paternain–Petean on geodesic flows. We use the geometric idea and the Floer homological technique from [**19**], but in addition apply the sandwiching method. Theorem 1 can be reformulated as follows.

THEOREM 1'. *The topological entropy of any Reeb flow on the spherization* SM *of* T^*M *is positive.*

For $q \in M$ abbreviate $\Sigma_q = \Sigma \cap T_q^*M$. The following corollary extends results of Morse and Gromov on the number of geodesics between two points.

COROLLARY 1. *Given* $q \in M$, *for almost every* $q' \in M$ *the number of orbits of the flow* φ^t *from* Σ_q *to* $\Sigma_{q'}$ *grows exponentially in time.*

In the lowest dimension, Theorem 1 yields the existence of many *closed* orbits.

COROLLARY 2. *Let* M *be a closed surface different from* S^2, $\mathbb{R}\mathrm{P}^2$, *the torus and the Klein bottle. Then* φ^t *carries a horseshoe. In particular, the number of geometrically distinct closed orbits grows exponentially in time.*

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1. *Introduction and main results*

The topological entropy $h_{\text{top}}(\varphi)$ of a diffeomorphism φ of a compact manifold P is a basic numerical invariant measuring the orbit structure complexity of φ. There are various ways of defining $h_{\text{top}}(\varphi)$, see [**26**] and Section 5·2 below. In this paper, we show that for a certain class of diffeomorphisms φ on a certain class of manifolds P, the number $h_{\text{top}}(\varphi)$ is always positive. We start with introducing the manifolds, first addressing the base manifolds M and then the hypersurfaces $\Sigma \subset T^*M$, and then describe the diffeomorphisms studied.

1·1. *Energy hyperbolic manifolds*

The complexity of our flows on $\Sigma \subset T^*M$ comes from the complexity of the loop space of the base manifold M. Let (M, g) be a C^∞-smooth closed Riemannian manifold. We assume throughout that M is connected. Fix $q_0 \in M$ and denote by $\Omega^1(M, q_0)$ the space of all paths $q\colon [0,1] \to M$ of Sobolev class $W^{1,2}$ such that $q(0) = q(1) = q_0$. This space has a canonical Hilbert manifold structure, [**28**]. The energy functional $\mathcal{E} = \mathcal{E}_g \colon \Omega^1(M, q_0) \to \mathbb{R}$ is defined as

$$\mathcal{E}(q) := \frac{1}{2}\int_0^1 |\dot q(t)|^2\, dt$$

where $|\dot q(t)|^2 = g_{q(t)}(\dot q(t), \dot q(t))$. For $a > 0$ we consider the sublevel sets

$$\mathcal{E}^a(q_0) := \left\{ q \in \Omega^1(M, q_0) \mid \mathcal{E}(q) \leqslant a \right\}.$$

Let $\mathbb{P}$ be the set of non-negative integers which are prime or 0, and write $\mathbb{F}_p = \mathbb{Z}_p$ and $\mathbb{F}_0 = \mathbb{Q}$. Throughout, H_* denotes singular homology. Let

$$\iota_k\colon \mathrm{H}_k\left(\mathcal{E}^a(q_0); \mathbb{F}_p\right) \longrightarrow \mathrm{H}_k\left(\Omega^1(M, q_0); \mathbb{F}_p\right)$$

be the homomorphism induced by the inclusion $\mathcal{E}^a(q_0) \hookrightarrow \Omega^1(M, q_0)$. It is well-known that for each a the homology groups $\mathrm{H}_k\left(\mathcal{E}^a(q_0); \mathbb{F}_p\right)$ vanish for all large enough k, cf. [**3**]. Therefore, the sums in the following definition are finite. Following [**19**] we give the following:

Definition. The Riemannian manifold (M, g) is *energy hyperbolic* if

$$C(M, g) := \sup_{p \in \mathbb{P}} \liminf_{m \to \infty} \frac{1}{m} \log \sum_{k \geqslant 0} \dim \iota_k \mathrm{H}_k(\mathcal{E}^{\frac{1}{2}m^2}(q_0); \mathbb{F}_p) > 0.$$

Remarks. Since M is connected, $C(M, g)$ does not depend on q_0. Since M is closed, the property "energy hyperbolic" does not depend on g. We say that the closed manifold M is *energy hyperbolic* if (M, g) is energy hyperbolic for some and hence any Riemannian metric g on M.

Examples. "Most" closed manifolds are energy hyperbolic:

(1) *Manifolds whose fundamental group has exponential growth.* Assume that the fundamental group $\pi_1(M)$ has exponential growth. This is, for instance, the case if M admits a Riemannian metric of negative sectional curvature, [**32**]. By [**19**, proposition 1·8], M is energy hyperbolic.

(2) *Hyperbolic manifolds.* Given a continuous map $f\colon (X, x) \to (Y, f(x))$ between path-connected pointed spaces, let $\Omega(f)\colon \Omega(X) \to \Omega(Y)$ be the induced map between based loop spaces and let

$$\mathrm{H}_*(\Omega(f); \mathbb{F}_p)\colon \mathrm{H}_*(\Omega(X); \mathbb{F}_p) \longrightarrow \mathrm{H}_*(\Omega(Y); \mathbb{F}_p)$$

be the map induced in homology. Following [**38**] we say that a closed connected manifold M is *hyperbolic* if there exists a finite simply connected CW complex K and a continuous map $f: K \to M$ such that for some $p \in \mathbb{P}$ the sequence

$$r_m(M, K, f; \mathbb{F}_p) := \sum_{k=0}^{m} \operatorname{rank} \mathrm{H}_k(\Omega(f); \mathbb{F}_p)$$

grows exponentially in m. In particular, rationally hyperbolic manifolds (such as $\mathbb{C}\mathrm{P}^2$ blown up in at least two points) are hyperbolic, [**38**]. An example of a hyperbolic manifold that is not rationally hyperbolic is $T^4 \# \overline{\mathbb{C}\mathrm{P}}^2$. By [**19**, proposition 1·11], hyperbolic manifolds are energy hyperbolic. We refer to [**19**] and the references therein for a thorough discussion of these two classes of energy-hyperbolic manifolds and for more examples.

1·2. *Fiberwise starshaped hypersurfaces in T^*M*

Let Σ be a smooth connected hypersurface in T^*M. We say that Σ is *fiberwise starshaped* if for each point $q \in M$ the set $\Sigma_q := \Sigma \cap T_q^*M$ is the smooth boundary of a domain in T_q^*M which is strictly starshaped with respect to the origin $0_q \in T^*M$. This means that the radial vector field $\sum_i p_i \, \partial p_i$ is transverse to each Σ_q. We assume throughout that $\dim M \geqslant 2$. Then $T^*M \setminus \Sigma$ has two components, the bounded inner part $\mathring{D}(\Sigma)$ containing the zero section, and the unbounded outer part $D^c(\Sigma) = T^*M \setminus D(\Sigma)$, where $D(\Sigma)$ denotes the closure of $\mathring{D}(\Sigma)$.

1·3. *Dynamics on fiberwise starshaped hypersurfaces*

Given a fiberwise starshaped hypersurface $\Sigma \subset T^*M$, choose a smooth function $H: T^*M \to \mathbb{R}$ such that $H^{-1}(1) = \Sigma$ and such that 1 is a regular value of H. Let

$$\omega = \sum_{i=1}^{d} dp_i \wedge dq_i \tag{1}$$

be the standard symplectic form on T^*M. The Hamiltonian vector field X_H defined by

$$\omega(X_H, \cdot) = -dH(\cdot) \tag{2}$$

defines a (local) flow φ_H^t on T^*M, called the Hamiltonian flow of H. It restricts to a flow $\varphi_H^t|_\Sigma$ on Σ. Our sign convention in (1) and (2) is such that the flow φ_H^t of a geodesic Hamiltonian $H = (1/2)|p|^2$ is the geodesic flow.

The orbits of $\varphi_H^t|_\Sigma$ do not depend on H in the sense that for a different choice G of the Hamiltonian function, the flow $\varphi_G^t|_\Sigma$ is a time change of the flow $\varphi_H^t|_\Sigma$, i.e., $\varphi_G^t|_\Sigma(x) = \varphi_H^{\sigma(t,x)}|_\Sigma(x)$ for a smooth positive function σ on $\mathbb{R} \times \Sigma$. For a flow without fixed points, any time change preserves vanishing of the topological entropy, see [**26**, p. 113]. Therefore $h_{\mathrm{top}}(\varphi_H^t|_\Sigma) > 0$ iff $h_{\mathrm{top}}(\varphi_G^t|_\Sigma) > 0$. Similarly, since Σ is compact, given $q, q' \in M$ the number of φ_H^t-orbits from Σ_q to $\Sigma_{q'}$ grows exponentially in time iff this is so for the number of φ_G^t-orbits.

1·4. *Main result*

THEOREM 1. *Consider an energy hyperbolic manifold M and a fiberwise starshaped hypersurface $\Sigma \subset T^*M$. Then $h_{\mathrm{top}}(\varphi_H^t|_\Sigma) > 0$.*

Discussion. (1) The unit cosphere bundle

$$S_1M(g) := \{(q, p) \in T^*M \mid |p| = 1\}$$

associated to a Riemannian metric g on M is an example of a fiberwise starshaped hypersurface. In this special case, Theorem 1 has been proved by Dinaburg [**9**], Gromov [**21**], Paternain [**35**, **36**] and Paternain–Petean [**38**] in their study of geodesic flows. More generally, the sets Σ_q are all convex if and only if Σ is the level set of a Finsler metric. In this case, Theorem 1 also follows from Paternain's work, [**36**].

(2) The assumption that M is energy hyperbolic obviously cannot be omitted: Geodesic flows over round spheres or flat tori have vanishing topological entropy. The assumption that Σ is fiberwise starshaped *with respect to the origin* cannot be omitted either: There are exact magnetic flows on compact quotients of **Sol** whose topological entropy drops to zero when the energy levels cease to inclose the zero section, see Section 7.

(3) Recall that a hypersurface $\Sigma \subset T^*M$ is said to be *of restricted contact type* if there exists a vector field X on T^*M such that $\mathcal{L}_X\omega = d\iota_X\omega = \omega$ and such that X is everywhere transverse to Σ, pointing outwards. Equivalently, there exists a 1-form α on T^*M such that $d\alpha = \omega$ and such that $\alpha \wedge (d\alpha)^{d-1}$ is a volume form on Σ orienting Σ as the boundary of $D(\Sigma)$. The correspondence is given by $\alpha = \iota_X\omega$. Our assumption that Σ is fiberwise starshaped translates to the assumption that Σ is of restricted contact type *with respect to* the Liouville vector field $Y = \sum_{i=1}^d p_i\,\partial p_i$, or, equivalently, the Liouville form

$$\lambda = \sum_{i=1}^{d} p_i\,dq_i.$$

defines a contact form on Σ.

QUESTION 1·1. Is Theorem 1 true for any hypersurface $\Sigma \subset T^*M$ of restricted contact type that encloses the 0-section?

1·5. *Reformulation of Theorem 1*

We are going to reformulate Theorem 1, which is formulated in terms of Hamiltonian dynamics, in a more invariant way by means of contact geometry. Let M be a closed connected manifold. Fix a fiberwise starshaped hypersurface $\Sigma \subset T^*M$. The hyperplane field $\xi_\Sigma = \ker(\lambda|_\Sigma) \subset T\Sigma$ is a contact structure on Σ. If Σ' is another fiberwise starshaped hypersurface, then the differential of the diffeomorphism

$$\Psi_{\Sigma\Sigma'}\colon \Sigma \longrightarrow \Sigma', \quad (q, p) \longmapsto (q, \psi(q, p)\,p),$$

obtained by fiberwise radial projection, maps ξ_Σ to $\xi_{\Sigma'}$ (because $\Psi^*_{\Sigma\Sigma'}(\lambda|_{\Sigma'}) = \psi\,\lambda|_\Sigma$), and is hence a contactomorphism $(\Sigma, \xi_\Sigma) \to (\Sigma', \xi_{\Sigma'})$. These contact manifolds can thus be identified, and are called the *spherization* (SM, ξ) of the cotangent bundle (T^*M, ω). [1]

Fix a representative (Σ, ξ_Σ). For every positive smooth function $f\colon \Sigma \to \mathbb{R}$ we still have $\xi_\Sigma = \ker(f\lambda|_\Sigma)$. The *Reeb vector field* R_f on $T\Sigma$ is defined as the unique vector field such that

$$d(f\lambda)(R_f, \cdot) \equiv 0, \qquad f\lambda(R_f) \equiv 1.$$

[1] The authors of [**10**] call it *space of oriented contact elements* and write $\mathbb{P}_+T^*M$. We have chosen the above wording since the unit cosphere bundle $(S_1M(g), \ker\lambda)$ with respect to a Riemannian metric g is a representative.

Its flow is called the *Reeb flow* of R_f. For $f \equiv 1$ the Reeb flow of R_f is a time change of the flow $\varphi_H^t|_\Sigma$ of any Hamiltonian function $H: T^*M \to \mathbb{R}$ with $H^{-1}(1) = \Sigma$ and such that 1 is a regular value. For different functions f the Reeb flows on Σ can be completely different, see e.g. [**23**].

Given another hypersurface Σ', let $\psi: \Sigma \to \mathbb{R}$ be the positive function such that

$$\Sigma' = \left\{ \big(q, \psi(q,p)\, p\big) \mid (q,p) \in \Sigma \right\}.$$

Then $d\Psi_{\Sigma\Sigma'}(R_\lambda) = R_{\psi^{-1}\lambda}$, that is, $\Psi_{\Sigma\Sigma'}$ conjugates the Reeb flows on $\big(\Sigma, \lambda\big)$ and $\big(\Sigma', \psi^{-1}\lambda\big)$. Summarizing, we have that the set of Reeb flows on (SM, ξ) is in bijection with Hamiltonian flows on fiberwise starshaped hypersurfaces in T^*M, up to time changes. Recall that for a flow without fixed points, any time change preserves vanishing of the topological entropy. Theorem 1 is therefore equivalent to:

THEOREM 1'. *The topological entropy of any Reeb flow on the spherization SM of T^*M is positive.*

1·6. *Uniform exponential growth of the number of Reeb chords*

Let M, Σ and H be as in Theorem 1. For $q, q' \in M$ let $\nu_T(q, q', H)$ be the number of flow lines of $\varphi_H^t|_\Sigma$ starting from Σ_q at $t = 0$ and arriving on $\Sigma_{q'}$ before time T. From our proof of Theorem 1 we shall immediately obtain

COROLLARY 1. *There exists a constant $h > 0$ depending only on (M, g), Σ and H such that for each $q \in M$,*

$$\liminf_{n \to \infty} \frac{1}{n} \log \nu_n(q, q', H) \geqslant h$$

for almost every $q' \in M$. Moreover, for every q' there exists an orbit of φ_H^t from Σ_q to $\Sigma_{q'}$.

For geodesic flows such lower bounds were obtained by Morse and Gromov [**21**]. Denote the fiber of SM over q by S_qM. In terms of Reeb flows, Corollary 1 reads:

COROLLARY 1'. *Fix a Reeb flow φ^t on (SM, ξ). There exists a constant $h > 0$ depending only on (M, g) and φ^t such that for each $q \in M$,*

$$\liminf_{n \to \infty} \frac{1}{n} \log \nu_n(q, q', \varphi^t) \geqslant h$$

for almost every $q' \in M$. Moreover, for every q' there exists a Reeb chord from S_qM to $S_{q'}M$.

Note that S_qM and $S_{q'}M$ are Legendrian submanifolds of (SM, ξ). Corollary 1' is thus a special case of the Arnold chord conjecture, with multiplicities.

Let (M, g) be a closed Riemannian manifold, endowed with its Riemannian measure μ_g, and let φ_g be the geodesic flow on $S_1M(g)$. It has been shown by Mañé [**31**] and Paternain–Paternain [**37**] that

$$h_{\mathrm{top}}(\varphi_g) = \lim_{T \to \infty} \frac{1}{T} \log \int_{M \times M} \nu_T(q, q')\, d\mu_g(q)\, d\mu_g(q'). \tag{3}$$

QUESTION 1·2. Is the identity (3) true for all Reeb flows on SM?

For a partial answer in the positive in the case of certain exact magnetic flows see [**34**].

In dimension 2, Theorem 1 yields the existence of many *closed* orbits. Indeed, according to [**24**] and [**25**, theorem 4·1], a smooth flow on a closed 3-manifold with positive topological entropy has exponential growth of closed orbits.

COROLLARY 2. *Let M be a closed surface different from S^2, $\mathbb{R}P^2$, the torus and the Klein bottle. Then φ_H^t carries a horseshoe. In particular, the number of geometrically distinct closed orbits grows exponentially in time.*

This is a special case of the Weinstein conjecture, with multiplicities. In terms of Reeb flows, Corollary 2 means that for such surfaces M, *for any Reeb flow on SM the number of geometrically distinct closed Reeb chords grows exponentially in time.* For a generalization of this result to generic Reeb flows on spherizations over higher dimensional closed manifolds M we refer to [**22**]. For a result on exponential growth rate of the number of closed orbits for certain Reeb flows on a large class of closed contact 3-manifolds, see [**8**].

Remark 1·3. In this paper we have exploited the *exponential* growth of the homology of the based loop space to obtain lower bounds for the entropy and the number of Reeb chords. For base manifolds M whose based loop space grows only polynomially, the same method yields lower bounds for the *slow entropy*, and *polynomial* lower bounds for the number of Reeb chords, see [**44**]. In particular, for any two points q, q' in a closed manifold M and any Reeb flow on SM there exists a Reeb chord from Σ_q to $\Sigma_{q'}$.

The paper is organized as follows. In the next section we give the idea of the proof. In Section 3 we define the relevant Hamiltonians and compute their action spectra. In Section 4 we recall Lagrangian Floer homology and compute it for our two Lagrangians in question. In Sections 5 and 6 we derive Theorem 1 and Corollary 1. Finally, in Section 7 we discuss the exact magnetic flows mentioned in Discussion 1·2.

2. *Idea of the proof*

Let M be an energy-hyperbolic manifold, and let $\Sigma \subset T^*M$ be a fiberwise starshaped hypersurface. Recall that $D(\Sigma)$ denotes the closure of the bounded part of $T^*M \setminus \Sigma$. For $q \in M$ let

$$D_q(\Sigma) = D(\Sigma) \cap T_q^*M$$

be the closed starshaped disc over q with boundary $\Sigma_q = \Sigma \cap T_q^*M$. Choose a smooth Hamiltonian function $K: T^*M \to \mathbb{R}$ that is fiberwise homogenous of degree 2, up to a smoothening near the zero section M. Recall that it suffices to show that $h_{\text{top}}(\varphi_K|_\Sigma) > 0$. Since K is fiberwise homogenous of degree 2, its Hamiltonian flows on the levels $s\Sigma$ agree up to constant time changes. Therefore, $h_{\text{top}}(\varphi_K|_\Sigma) > 0$ is equivalent to $h_{\text{top}}(\varphi_K|_{D(\Sigma)}) > 0$. By a theorem of Yomdin, the latter inequality will follow if we can find a point $q \in M$ such that the volume of the discs $\varphi_K^n\big(D_q(\Sigma)\big)$ grows exponentially with n.

We shall, in fact, establish this for each point $q \in M$, by taking up an idea from [**18**]: Fix $q \in M$. By sandwiching the set Σ between two sphere bundles (which are the levels of a geodesic Hamiltonian), and by using the Lagrangian Floer homology of two fibers in T^*M and its isomorphism to the homology of the based loop space of M (whose dimensions grow exponentially by assumption), we shall show that for almost every other point $q' \in M$ the number of intersections

$$\varphi_K^n(D_q(\Sigma)) \cap D_{q'}(\Sigma)$$

grows exponentially with m. This means that the discs $\varphi_K^n(D_q(\Sigma))$ "wrap" exponentially often in n around the base M, and hence their volume grows exponentially.

Remark 2·1. As pointed out to us by Will Merry and Gabriel Paternain, our proof of Theorem 1 can be considerably shortened by using symplectic homology with its wrapped version for Lagrangian boundary conditions and a remark of Seidel in [**42**, section 4a]. Furthermore, it is conceivable that Theorem 1 (or at least Corollary 1) can also be obtained by using Contact homology, for spherizations, relative to two Legendrian submanifolds Σ_q and $\Sigma_{q'}$, or by a relative version of Rabinowitz–Floer homology, and by establishing an isomorphism between either of these homologies with the homology of the based loop space. This would have the advantage of working directly at the level Σ, instead of the sublevel set $D(\Sigma)$. These homologies are, however, not yet fully established. We thus preferred to work with Lagrangian Floer homology and to pass through the sublevel set $D(\Sigma)$.

3. *Relevant Hamiltonians and their action spectra*

3·1. *The path space* $\Omega^1(T^*M, q_0, q_1)$

For $q_0, q_1 \in M$ let $\Omega^1(T^*M, q_0, q_1)$ be the space of all paths $x\colon [0,1] \to T^*M$ of Sobolev class $W^{1,2}$ such that $x(0) \in T^*_{q_0}M$ and $x(1) \in T^*_{q_1}M$. This space has a canonical Hilbert manifold structure, [**28**]. Consider a proper Hamiltonian function $H\colon T^*M \to \mathbb{R}$; then its Hamiltonian flow φ_H^t exists for all times. The action functional of classical mechanics $\mathcal{A}_H\colon \Omega^1(T^*M, q_0, q_1) \to \mathbb{R}$ associated with H is defined as

$$\mathcal{A}_H(x) = \int_0^1 (\lambda(\dot{x}(t)) - H(x(t)))dt, \tag{4}$$

where $\lambda = \sum_{j=1}^d p_j\, dq_j$ is the canonical 1-form on T^*M. This functional is C^∞-smooth, and its critical points are precisely the elements of the space $\mathcal{P}(H, q_0, q_1)$ of C^∞-smooth paths $x\colon [0,1] \to T^*M$ solving

$$\dot{x}(t) = X_H(x(t)),\ \ t \in [0,1], \qquad x(j) \in T^*_{q_j}M,\ \ j = 0, 1.$$

Notice that the elements of $\mathcal{P}(H, q_0, q_1)$ correspond to the intersection points of $\varphi_H(T^*_{q_0}M)$ and $T^*_{q_1}M$ via the evaluation map $x \mapsto x(1)$.

3·2. *The Hamiltonians* $G_- \leqslant K \leqslant G_+$

Since Σ is fiberwise starshaped, we can define a function $F\colon T^*M \to \mathbb{R}$ by the two requirements

$$F|_\Sigma \equiv 1, \qquad F(q, sp) = s^2 F(q, p) \quad \text{for all } s \geqslant 0 \text{ and } (q,p) \in T^*M.$$

This function is fiberwise homogenous of degree 2, of class C^1, and smooth off the zero-section. To smoothen F, choose $\varepsilon \in (0, 1/4)$. We shall appropriately fix ε later on. Choose a smooth function $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} f(r) = 0 & \text{if } r \leqslant \varepsilon^2, \\ f(r) = r & \text{if } r \geqslant \varepsilon, \\ f'(r) > 0 & \text{if } r > \varepsilon^2, \\ 0 \leqslant f'(r) \leqslant 2 & \text{for all } r, \end{cases} \tag{5}$$

see Figure 1.

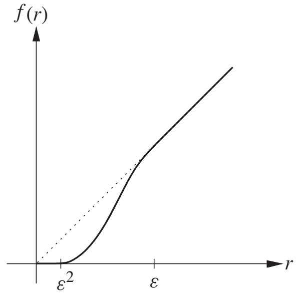

Fig. 1. The "cut off" function f.

Then $f \circ F$ is smooth. The Lagrangian Floer homology of $f \circ F$ (with respect to two fibers of T^*M) is, in general, not defined, since the moduli spaces of Floer trajectories may not be compact. We shall therefore alter $f \circ F$ near infinity to a Riemannian Hamiltonian. We denote canonical coordinates on T^*M by (q, p). Fix a Riemannian metric g on M, and let $|\cdot|$ be the norm on the fibers of T^*M induced by g. Define $G(q, p) = (1/2)\,|p|^2$. Recall that M is compact and that Σ is fiberwise starshaped. After multiplying g by a positive constant, we can assume that $G \leqslant F$. Choose $\sigma > 0$ such that $\sigma G \geqslant F$.

For $r > 0$ we abbreviate

$$D(r) = \{(q, p) \in T^*M \mid |p| \leqslant r\}.$$

Note that $D(2) = \{G \leqslant 2\}$. Choose a smooth function $\tau : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} \tau(r) &= 0 \quad \text{if} \quad r \leqslant 2, \\ \tau(r) &= 1 \quad \text{if} \quad r \geqslant 4, \\ \tau'(r) &\geqslant 0 \quad \text{for all } r \in \mathbb{R}. \end{aligned}$$

Define

$$G_+ = \sigma G \tag{6}$$

$$K(q, p) = (1 - \tau(|p|))(f \circ F)(q, p) + \tau(|p|)G_+(q, p), \tag{7}$$

$$G_-(q, p) = (1 - \tau(|p|))(f \circ G)(q, p) + \tau(|p|)G_+(q, p). \tag{8}$$

Then

$$G_- \leqslant K \leqslant G_+ \tag{9}$$

and

$$K = f \circ F \quad \text{and} \quad G_- = f \circ G \quad \text{on } D(2). \tag{10}$$

Since $D(\Sigma) = \{F \leqslant 1\} \subset \{F \leqslant 2\} \subset \{G_- \leqslant 2\}$, we in particular have

$$K = f \circ F \quad \text{on } D(\Sigma). \tag{11}$$

Moreover,

$$G_- = K = G_+ \quad \text{outside } D(4). \tag{12}$$

3·3. *Action spectra of nG_-, nK, nG_+*

We next investigate the action spectra of the functions nG_-, nK, nG_+ for $n \in \mathbb{N}$. The action spectrum $\mathcal{S}(H, q_0, q_1)$ of a proper Hamiltonian $H : T^*M \to \mathbb{R}$ is the set of critical

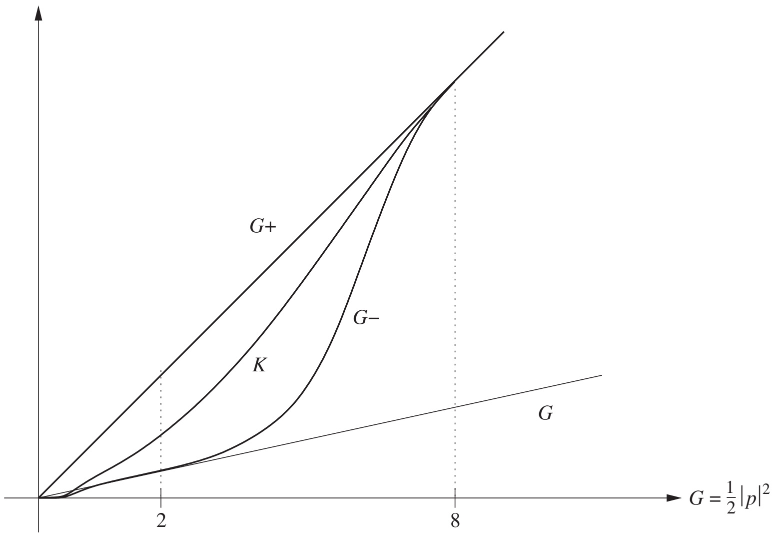

Fig. 2. The functions $G_- \leqslant K \leqslant G_+$, schematically.

values of $\mathcal{A}_H : \Omega^1(T^*M, q_0, q_1) \to \mathbb{R}$,

$$\mathcal{S}(H, q_0, q_1) := \{\mathcal{A}_H(x) \mid x \in \mathcal{P}(H, q_0, q_1)\}.$$

We first look at the action spectra of Hamiltonians homogenous of degree 2.

LEMMA 3·1. *Let $H : T^*M \to \mathbb{R}$ be fiberwise homogenous of degree 2, let $h : \mathbb{R} \to \mathbb{R}$ be a smooth function, and let $c > 0$.*

(i) *For $\gamma \in \mathcal{P}(h \circ H, q_0, q_1)$ we have*

$$\mathcal{A}_{h\circ H}(\gamma) = 2h'(H(\gamma))H(\gamma) - h(H(\gamma)).$$

(ii) $\mathcal{S}(cH, q_0, q_1) = (1/c)\mathcal{S}(H, q_0, q_1)$.

Proof. (i) Let $Y = \sum_i p_i\, \partial p_i$. For each $t \in [0, 1]$ we have $\dot\gamma(t) = X_{h\circ H}(\gamma(t))$, and hence

$$\lambda(\dot\gamma(t)) = \omega(Y, X_{h\circ H}(\gamma(t))) = d(h\circ H)(\gamma(t))(Y) = h'(H(\gamma(t)))dH(\gamma(t))(Y).$$

Since H is fiberwise homogenous of degree 2, Euler's identity yields

$$dH(\gamma(t))(Y) = 2H(\gamma(t))$$

whence

$$\lambda(\dot\gamma(t)) = 2h'(H(\gamma(t))H(\gamma(t)).$$

Therefore,

$$\mathcal{A}_{h\circ H}(\gamma) = \int_0^1 (\lambda(\dot\gamma(t)) - h(H(\gamma(t))))dt = 2h'(H(\gamma))H(\gamma) - h(H(\gamma)),$$

as claimed.

(ii) By definition (2) we have $X_{cH}(q, p) = cX_H(q, p)$, and since H is homogenous of degree 2, we have $X_H(q, (1/c)p) = (1/c)X_H(q, p)$ (where in the latter equality we have canonically identified $T_{(q, \frac{1}{c}p)}T^*M = T_{(q,p)}T^*M$)). Therefore,

$$X_{cH}(q, (1/c)p) = cX_H(q, (1/c)p) = X_H(q, p).$$

To $\gamma(t) = (q(t), p(t))$ in $\mathcal{P}(H, q_0, q_1)$ therefore corresponds $\gamma_c(t) = (q(t), (1/c)p(t))$ in $\mathcal{P}(cH, q_0, q_1)$. Assertion (i) now yields

$$\mathcal{A}_{cH}(\gamma_c) = cH(\gamma_c) = cH(q(0), (1/c)p(0)) = (1/c)H(q(0), p(0)) = (1/c)\mathcal{A}_H(\gamma),$$

and so $\mathcal{S}(cH, q_0, q_1) = (1/c)\mathcal{S}(H, q_0, q_1)$.

We next look at the action spectrum of nK.

PROPOSITION 3·2. *Let $\gamma \in \mathcal{P}(nK, q_0, q_1)$. If $\gamma \subset \mathring{D}(\Sigma)$, then $\mathcal{A}_{nK}(\gamma) < n$. If $\gamma \subset D^c(\Sigma)$, then $\mathcal{A}_{nK}(\gamma) > n$.*

Proof. Assume first that $\gamma \subset \mathring{D}(\Sigma)$. Then $F(\gamma) < 1$ and $\gamma \in \mathcal{P}(n(f \circ F), q_0, q_1)$ by (11). Hence Lemma 3·1 (i) and our choice of f yield

$$\mathcal{A}_{nK}(\gamma) = \mathcal{A}_{n(f\circ F)}(\gamma) = 2nf'(F(\gamma))F(\gamma) - nf(F(\gamma)) < n.$$

Assume now that $\gamma \subset D^c(\Sigma)$. Then $F(\gamma) > 1$ and $n(f \circ F)(\gamma) = nF(\gamma)$. Let again $Y = \sum_i p_i \partial p_i$. For $t \in [0, 1]$ we compute at $\gamma(t) = (q, p)$, using (7),

$$\begin{aligned} d(nK)(\gamma(t))(Y) = &-\tau'(|p|)nF(q, p)|p| + (1 - \tau(|p|))\, d(nF)(q, p)(Y) \\ &+ \tau'(|p|)nG_+(q, p)|p| + \tau(|p|)\, d(nG_+)(q, p)(Y). \end{aligned}$$

Since $\tau'(|p|) \geqslant 0$ and $nG_+(q, p) \geqslant nF(q, p)$, the sum of the first and third summand is $\geqslant 0$. Together with Euler's identity applied to the functions nF and nG_+ fiberwise homogenous of degree 2 we obtain

$$\begin{aligned} d(nK)(\gamma(t))(Y) - nK(\gamma(t)) &\geqslant (1 - \tau(|p|))(d(nF)(\gamma(t))(Y) - nF(\gamma(t))) \\ &\quad + \tau(|p|)(d(nG_+)(\gamma(t))(Y) - nG_+(\gamma(t))) \\ &= (1 - \tau(|p|))\, nF(\gamma(t)) + \tau(|p|)nG_+(\gamma(t)) \\ &\geqslant nF(\gamma(t)) \\ &> n. \end{aligned}$$

Hence $\mathcal{A}_{nK}(\gamma) = \int_0^1 (d(nK)(\gamma(t))(Y) - nK(\gamma(t)))dt > n$, as claimed.

3·3·1. *The Non-crossing lemma*

For $n \in \mathbb{N}$ consider the space of Hamiltonian functions

$$\mathcal{H}_4(nG_+) = \{H : T^*M \to \mathbb{R} \mid H = nG_+ \text{ on } T^*M \setminus D(4)\}.$$

Note that nG_-, nK, nG_+ belong to $\mathcal{H}_4(nG_+)$. For $a \in \mathbb{R}$ define

$$\mathcal{H}_4^a(nG_+) = \{H \in \mathcal{H}_4(nG_+) \mid a \notin \mathcal{S}(H, q_0, q_1)\}.$$

Fix a smooth function $\beta : \mathbb{R} \to [0, 1]$ such that

$$\beta(s) = 0 \text{ for } s \leqslant 0, \quad \beta(s) = 1 \text{ for } s \geqslant 1, \quad \beta'(s) \geqslant 0 \text{ for all } s \in \mathbb{R}. \tag{13}$$

For $s \in [0, 1]$ define the functions

$$G_s = (1 - \beta(s))G_- + \beta(s)G_+. \tag{14}$$

Then $nG_s \in \mathcal{H}_4(nG_+)$ for each $n \in \mathbb{N}$ and $s \in [0, 1]$.

Recall that $\sigma \geqslant 1$. Choose $a \in (n, n + 1)$ and define the function $a(s) : [0, 1] \to \mathbb{R}$ by

$$a(s) = \frac{a}{1 + \beta(s)(\sigma - 1)}. \tag{15}$$

Note that $a(s)$ is monotone decreasing with minimum $a(1) = a/\sigma$. We assume from now on that $\varepsilon \in (0, 1/4)$ entering the definition of the smoothing function f also meets

$$\varepsilon^2 < \frac{1}{2\sigma}. \tag{16}$$

LEMMA 3·3. *If $a \notin \mathcal{S}(nG, q_0, q_1)$, then $a(s) \notin \mathcal{S}(nG_s, q_0, q_1)$ for $s \in [0, 1]$.*

Proof. Take $\gamma \in \mathcal{P}(nG_s, q_0, q_1)$. In view of (8) and (14), the orbit γ lies on a level set of G.

Assume first that $\gamma \in \mathcal{P}(nG_s)$ lies outside $D(2)$. Then $f \circ G(\gamma) = G(\gamma)$, so that

$$nG_s \;=\; n(1 - \beta(s))((1 - \tau(|p|))G + \tau(|p|)G_+) + n\beta(s)G_+.$$

Computing as in the proof of Proposition 3·2 (with F replaced by G) we find

$$\begin{aligned} d(nG_s)(\gamma(t))(Y) - nG_s(\gamma(t)) &\geqslant n(1 - \beta(s))((1 - \tau(|p|))G(\gamma(t)) + \tau(|p|)G_+(\gamma(t))) \\ &\quad\; + n\beta(s)G_+(\gamma(t)) \\ &\geqslant n(1 - \beta(s))(G(\gamma(t))) + n\beta(s)G(\gamma(t)) \\ &= nG(\gamma(t)) \\ &\geqslant 2n. \end{aligned}$$

Hence $\mathcal{A}_{nG_s}(\gamma) \geqslant 2n \geqslant n + 1 > a \geqslant a(s)$.

Assume next that γ lies in $D(\varepsilon)$. Then $\tau(|p(\gamma(t))|) = 0$ for all $t \in [0, 1]$. Hence $G_-(\gamma) = f \circ G(\gamma)$ and

$$G_s(\gamma) = (1 - \beta(s))(f \circ G)(\gamma) + \beta(s)\sigma G(\gamma).$$

Applying Lemma 3·1 (i) with $h = \big(1 - \beta(s)\big) f + \beta(s)\sigma$ we therefore find

$$\mathcal{A}_{nG_s}(\gamma) = 2n((1 - \beta(s)) f'(G(\gamma)))G(\gamma) - nG_s(\gamma).$$

Together with $f' \leqslant 2$ and (16) we obtain

$$\mathcal{A}_{nG_s}(\gamma) \leqslant 4nG(\gamma) \leqslant 2n\varepsilon^2 < \frac{n}{\sigma} \leqslant a(1) \leqslant a(s).$$

Assume finally that γ lies in $D(2) \setminus D(\varepsilon)$. Then $\tau(|p(\gamma)|) = 0$ and $f \circ G(\gamma) = G(\gamma)$. Hence $G_-(\gamma) = G(\gamma)$ and

$$\begin{aligned} nG_s(\gamma) &= n((1 - \beta(s))G + \beta(s)\sigma G_-)(\gamma) \\ &= (1 + \beta(s)(\sigma - 1))nG(\gamma). \end{aligned}$$

If $\mathcal{A}_{nG_s}(\gamma) = a(s)$, then $a = a(0) \in \mathcal{S}(nG, q_0, q_1)$ in view of Lemma 3·1 (ii) and the definition (15) of $a(s)$, contradicting our hypothesis.

3·4. *Almost all fibers are good*

Fix now $q_0 \in M$. For $q \in M$ define

$$D_q(4) = \{p \in T_q^*M \mid |p| \leqslant 4\} = D(4) \cap T_q^*M.$$

Fix $n \in \mathbb{N}$. Given $H \in \mathcal{H}_4(nG_+)$ let $U(q_0, H)$ be the set of those $q_1 \in M$ for which $\varphi_H(D_{q_0}(4))$ and $D_{q_1}(4)$ intersect transversely. Let μ_g be the Riemannian measure on (M, g). We say that a set $U \subset M$ has *full measure* if $\mu_g(U) = \mu_g(M)$. Note that this property does not depend on the Riemannian metric g.

LEMMA 3·4. *The set $U(q_0, H)$ is open and of full measure in M.*

Proof. Since $D_{q_0}(4)$ is compact, $U(q_0, H)$ is open. Applying Sard's Theorem to the projection $\varphi_H(D_{q_0}(4)) \to M$ one sees that $U(q_0, H)$ has full measure in M.

Note that for each $n \in \mathbb{N}$ we have $\varphi_{nH} = \varphi_H^n$. Let G_-, K, G_+ be the Hamiltonians defined in (8), (7), (6). Define

$$V_n(q_0) := U(q_0, nG_-) \bigcap U(q_0, nK) \bigcap U(q_0, nG_+).$$

For $q_1 \in V_n(q_0)$ all the sets $\varphi^n_{G_-}(D_{q_0}(4))$, $\varphi^n_K(D_{q_0}(4))$, $\varphi^n_{G_+}(D_{q_0}(4))$ intersect $D_{q_1}(4)$ transversely. Moreover,

COROLLARY 3·5. *The sets $V_n(q_0)$ have full measure in M.*

4. *Lagrangian Floer homology*

Floer homology for Lagrangian intersections was invented by Floer in a series of seminal papers, [**11–14**]. We shall use a version of Lagrangian Floer homology described in [**18**, **19**, **27**].

4·1. *Definition of* $\mathrm{HF}_*^a(H, q_0, q_1; \mathbb{F}_p)$

4·1·1. *The chain groups*

Let $H \in \mathcal{H}_4(nG_+)$, let $q_0 \in M$, and fix $q_1 \in U(q_0, H)$. For $a < n + 1$ define

$$\mathcal{P}^a(H, q_0, q_1) := \{x \in \mathcal{P}(H, q_0, q_1) \mid \mathcal{A}_H(x) < a\}.$$

For $\gamma \in \mathcal{P}(H, q_0, q_1)$ outside $D(4)$ we have, by Lemma 3·1 (i),

$$\mathcal{A}_H(\gamma) = nG_+(\gamma) = n\sigma G(\gamma) \geqslant 2n \geqslant n + 1$$

whence

$$\mathcal{P}^a(H, q_0, q_1) \subset D(4). \tag{17}$$

Since $q_1 \in U(q_0, H)$ we conclude that $\mathcal{P}^a(H, q_0, q_1)$ is a finite set. The fibers of T^*M form a Lagrangian foliation. For each path $x \in \mathcal{P}(H, q_0, q_1)$ the Maslov index $\mu(x)$ is therefore a well-defined integer. In case of a Riemannian Hamiltonian, $\mu(x)$ agrees with the Morse index of the corresponding geodesic path, see [**2**, section 1·2]. Define the kth Floer chain group $\mathrm{CF}_k^a(H, q_0, q_1; \mathbb{F}_p)$ as the finite-dimensional $\mathbb{F}_p$-vector space freely generated by the elements of $\mathcal{P}^a(H, q_0, q_1)$ of Maslov index k, and define the full Floer chain group as

$$\mathrm{CF}_*^a(H, q_0, q_1; \mathbb{F}_p) = \bigoplus_{k \in \mathbb{Z}} \mathrm{CF}_k^a(H, q_0, q_1; \mathbb{F}_p).$$

4·1·2. *Almost complex structures*

Recall that an almost complex structure J on T^*M is *ω-compatible* if

$$\langle \cdot, \cdot \rangle \equiv g_J(\cdot, \cdot) := \omega(\cdot, J\cdot)$$

defines a Riemannian metric on T^*M. Denote by $\xi = \ker(\iota_Y \omega|_{\partial D(4)})$ the contact structure on $\partial D(4)$ defined by Y and ω, and for $t \geqslant 0$ let ψ_t be the semi-flow of the Liouville vector field $Y = \sum p_j\, \partial p_j$ on $T^*M \setminus \mathring{D}(4)$. An ω-compatible almost complex structure J on T^*M is *convex* on $T^*M \setminus \mathring{D}(4)$ if

$$J\xi = \xi, \qquad \omega(Y(x), J(x)Y(x)) = 1, \qquad D\psi_t(x)J(x) = J(\psi_t(x))D\psi_t(x)$$

for all $x \in \partial D(4)$ and $t \geqslant 0$. Following [**4**, **7**, **43**] we consider the set $\mathcal{J}$ of t-dependent smooth families $\mathbf{J} = \{J_t\}$, $t \in [0, 1]$, of ω-compatible almost complex structures on T^*M such that J_t is convex and independent of t on $T^*M \setminus \mathring{D}(4)$. The set $\mathcal{J}$ is non-empty and connected.

4·1·3. *The boundary operators*

For $\mathbf{J} \in \mathcal{J}$, for smooth maps u from the strip $S = \mathbb{R} \times [0, 1]$ to T^*M, and for $x^\pm \in \mathcal{P}^a(H, q_0, q_1)$ consider the partial differential equation with boundary conditions

$$\begin{cases} \partial_s u - J_t(u)\,(\partial_t u - X_H(u)) = 0, \\ u(s, j) \in T^*_{q_j}M, \quad j = 0, 1, \\ \lim\limits_{s \to \pm\infty} u(s, t) \;=\; x^\pm(t) \;\text{ uniformly in } t. \end{cases} \tag{18}$$

The Cauchy–Riemann equation in (18) is called *Floer's equation*.

LEMMA 4·1. *Solutions of equation* (18) *are contained in* $D(4)$.

Sketch of proof. By (17) we have $x^\pm \subset D(4)$, whence

$$\lim_{s \to \pm\infty} u(s, t) \;=\; x^\pm(t) \subset D(4). \tag{19}$$

In view of the strong maximum principle, the lemma follows from the convexity of J outside $D(4)$ and from (19) together with the fact that $H = nG_+$ outside $D(4)$ implies $\omega\,(Y, JX_H) = 0$, cf. [**18**, **27**].

We denote the set of solutions of (18) by $\mathcal{M}(x^-, x^+, H; \mathbf{J})$. Note that the group $\mathbb{R}$ freely acts on $\mathcal{M}(x^-, x^+, H; \mathbf{J})$ by time-shift. Lemma 4·1 is an important ingredient to establish the compactness of the quotients $\mathcal{M}(x^-, x^+, H; \mathbf{J})/\mathbb{R}$. The other ingredient is that there is no bubbling-off of $\mathbf{J}$-holomorphic spheres or discs. Indeed, $[\omega]$ vanishes on $\pi_2(T^*M)$ because $\omega = d\lambda$ is exact, and $[\omega]$ vanishes on $\pi_2(T^*M, T^*_{q_j}M)$ because λ vanishes on $T^*_{q_j}M$, $j = 0, 1$. See for instance [**12**] or [**39**] for more details.

There exists a residual subset $\mathcal{J}_{\text{reg}}(H)$ of $\mathcal{J}$ such that for each $\mathbf{J} \in \mathcal{J}_{\text{reg}}(H)$ the linearized operator for Floer's equation is surjective for each solution of (18). For such a *regular* $\mathbf{J}$ the moduli space $\mathcal{M}\left(x^-, x^+, H; \mathbf{J}\right)$ is a smooth manifold of dimension $\mu(x^-) - \mu(x^+)$ for all $x^\pm \in \mathcal{P}^a(H, q_0, q_1)$, see [**17**]. Fix $\mathbf{J} \in \mathcal{J}_{\text{reg}}(H)$. It is shown in [**2**, section 1·4] that the manifolds $\mathcal{M}\left(x^-, x^+, H; \mathbf{J}\right)$ can be oriented in a way which is coherent with gluing. For $x^\pm \in \mathcal{P}^a(H, q_0, q_1)$ with $\mu(x^-) = \mu(x^+) + 1$ let

$$n(x^-, x^+, H; \mathbf{J}) \;=\; \#\mathcal{M}(x^-, x^+, H; \mathbf{J})/\mathbb{R} \in \mathbb{Z}$$

be the oriented count of the finite set $\mathcal{M}(x^-, x^+, H; \mathbf{J})/\mathbb{R}$. If $u \in \mathcal{M}(x^-, x^+, H; \mathbf{J})$, then $\mathcal{A}_H(x^-) \geqslant \mathcal{A}_H(x^+)$, see the more general Lemma 4·2 below. For $k \in \mathbb{Z}$ one can therefore define the Floer boundary operator

$$\partial_k(\mathbf{J})\colon\; \mathrm{CF}^a_k(H, q_0, q_1; \mathbb{F}_p) \longrightarrow \mathrm{CF}^a_{k-1}(H, q_0, q_1; \mathbb{F}_p)$$

as the linear extension of

$$\partial_k(\mathbf{J})\, x^- \;=\; \sum n(x^-, x^+, H; \mathbf{J})\, x^+$$

where $x^- \in \mathcal{P}^a(H, q_0, q_1)$ has index $\mu(x^-) = k$ and the sum runs over all $x^+ \in \mathcal{P}^a(H, q_0, q_1)$ of index $\mu(x^+) = k-1$. Then $\partial_{k-1}(\mathbf{J}) \circ \partial_k(\mathbf{J}) = 0$ for each k. The proof makes use of the compactness of the 0- and 1-dimensional components of $\mathcal{M}(x^-, x^+, H; \mathbf{J})/\mathbb{R}$, see [**2**, **11**, **41**]. As our notation suggests, the Floer homology groups

$$\mathrm{HF}_k^a(H, q_0, q_1; \mathbb{F}_p) := \ker \partial_k(\mathbf{J}) / \operatorname{im} \partial_{k+1}(\mathbf{J})$$

do not depend on the choices involved in their construction: They neither depend on coherent orientations up to canonical isomorphisms, [**2**, section 1·7], nor do they depend on $\mathbf{J} \in \mathcal{J}_{\mathrm{reg}}(H)$ up to natural isomorphisms, as a continuation argument shows, [**11**, **41**]. The groups $\mathrm{HF}_k^a(H, q_0, q_1; \mathbb{F}_p)$ do depend, however, on $a < n+1$ and $H \in \mathcal{H}_4(nG_+)$. In the sequel, the points $q_0, q_1 \in M$ and the field $\mathbb{F}_p$ are fixed throughout. We shall often write $\mathrm{CF}_*^a(H)$ and $\mathrm{HF}_*^a(H)$ instead of $\mathrm{CF}_*^a(H, q_0, q_1; \mathbb{F}_p)$ and $\mathrm{HF}_*^a(H, q_0, q_1; \mathbb{F}_p)$.

4·2. *Continuation homomorphisms*

The goal of this section is to relate the groups

$$\mathrm{HF}_*^a(nG_-), \quad \mathrm{HF}_*^a(nK), \quad \mathrm{HF}_*^a(nG_+).$$

Let $\beta\colon \mathbb{R} \to [0, 1]$ be the function from (13). Given two functions $H^-, H^+ \in \mathcal{H}_4(nG_+)$ with $H^-(x) \leqslant H^+(x)$ for all $x \in T^*M$ we form the monotone homotopy

$$H_s(x) = H^-(x) + \beta(s)(H^+(x) - H^-(x)). \tag{20}$$

Then $H_s \in \mathcal{H}_4(nG_+)$ for each s, and $H_s = H^-$ for $s \leqslant 0$ and $H_s = H^+$ for $s \geqslant 1$. Consider the equation with boundary conditions

$$\begin{cases} \partial_s u - J_{s,t}(u)(\partial_t u - X_{H_s}(u)) = 0, \\ u(s, j) \in T_{q_j}^* M, \;\; j = 0, 1, \\ \lim\limits_{s\to\pm\infty} u(s, t) = x^\pm(t) \;\; \text{uniformly in } t, \end{cases} \tag{21}$$

where $s \mapsto \{J_{s,t}\}$, $s \in \mathbb{R}$, $t \in [0, 1]$, is a *regular homotopy* of families $\{J_t\}$ of almost complex structures on T^*M. This means that:

(i) $J_{s,t}$ is ω-compatible and convex and independent of s and t outside $D(4)$;
(ii) $J_{s,t} = J_t^- \in \mathcal{J}_{\mathrm{reg}}(H^-)$ for $s \leqslant 0$;
(iii) $J_{s,t} = J_t^+ \in \mathcal{J}_{\mathrm{reg}}(H^+)$ for $s \geqslant 1$;
(iv) the solutions of (21) are transverse (that is, the associated Fredholm operators are surjective) and therefore form finite dimensional moduli spaces.

The following lemma is well known. We reprove it in view of the many different sign conventions (for the Hamilton equation, the action functional, the Floer equation, etc.) used by different authors.

LEMMA 4·2. *Assume that* $u\colon S = \mathbb{R} \times [0, 1] \to T^*M$ *is a solution of* (21). *Then*

$$\mathcal{A}_{H^+}(x^+) \leqslant \mathcal{A}_{H^-}(x^-) - \int_0^1 \int_{-\infty}^{\infty} \beta'(s)\,(H_1 - H_0)\,(u(s, t))\,ds\,dt.$$

Proof. Note that $\lambda = p\,dq$ vanishes along the Lagrangian boundary conditions $u(s, j) \in T_{q_j}^* M$ in (21). Since $\omega = d\lambda$ is exact, we obtain, using Stokes Theorem and taking into account orientations,

$$\int_S u^*\omega = \int_{u(\partial S)} \lambda = \int_{x^+} \lambda - \int_{x^-} \lambda.$$

Moreover, by (20),

$$\frac{d}{ds} H_s(u(s,t)) = dH_s(u(s,t))(\partial_s u) + \beta'(s)(H_1 - H_0)(u(s,t)).$$

This and the asymptotic boundary condition in (21) yield

$$\int_0^1 \int_{-\infty}^{\infty} dH_s(u(s,t))(\partial_s u)\, ds\, dt = \int_0^1 H^+(x^+)\, dt - \int_0^1 H^-(x^-)\, dt \\ - \int_0^1 \int_{-\infty}^{\infty} \beta'(s)(H_1 - H_0)(u(s,t))\, ds\, dt.$$

Together with the compatibility $g_{s,t}(v, w) = \omega(v, J_{s,t} w)$, Floer's equation in (21), Hamilton's equation (2) and the definition (4) of the action functional we obtain

$$\begin{aligned} 0 &\leqslant \int_0^1 \int_{-\infty}^{\infty} g_{s,t}(\partial_s u, \partial_s u)\, ds\, dt \\ &= \int_0^1 \int_{-\infty}^{\infty} g_{s,t}(\partial_s u, J_{s,t}(u)(\partial_t u - X_{H_s}(u)))\, ds\, dt \\ &= -\int_S u^* \omega - \int_0^1 \int_{-\infty}^{\infty} \omega\left(X_{H_s}(u), \partial_s u\right) ds\, dt \\ &= \int_{x^-} \lambda - \int_{x^+} \lambda + \int_0^1 \int_{-\infty}^{\infty} dH_s(u)(\partial_s u)\, ds\, dt \\ &= \int_{x^-} \lambda - \int_{x^+} \lambda + \int_0^1 H^+(x^+)\, dt - \int_0^1 H^-(x^-)\, dt \\ &\quad - \int_0^1 \int_{-\infty}^{\infty} \beta'(s)(H_1 - H_0)(u(s,t))\, ds\, dt \\ &= \mathcal{A}_{H^-}(x^-) - \mathcal{A}_{H^+}(x^+) - \int_0^1 \int_{-\infty}^{\infty} \beta'(s)(H_1 - H_0)(u(s,t))\, ds\, dt, \end{aligned}$$

as claimed.

In view of Lemma 4·2 the action decreases along solutions u of (21). By counting these solutions one can therefore define the Floer chain map

$$\phi_{H^+H^-}\colon \mathrm{CF}_*^a(H^-) \to \mathrm{CF}_*^a(H^+),$$

see [**7**, **15**, **16**, **40**]. The induced *continuation homomorphism*

$$\Phi_{H^+H^-}\colon \mathrm{HF}_*^a(H^-) \to \mathrm{HF}_*^a(H^+)$$

on Floer homology does not depend on the choice of the regular homotopy $\{J_{s,t}\}$ used in its definition. An important property of these homomorphisms is naturality with respect to concatenation,

$$\Phi_{H_3H_2} \circ \Phi_{H_2H_1} = \Phi_{H_3H_1} \quad \text{for } H_1 \leqslant H_2 \leqslant H_3. \tag{22}$$

Another important fact is the following invariance property, which is proved in [**7**, **16**] and [**4**, section 4·5].

LEMMA 4·3. *If $a \notin \mathcal{S}(H_s, q_0, q_1)$ for all $s \in [0,1]$, then $\Phi_{H^+H^-}\colon \mathrm{HF}_*^a(H^-) \to \mathrm{HF}_*^a(H^+)$ is an isomorphism.*

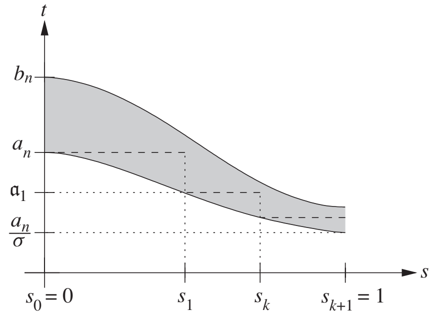

Fig. 3. The curves $a_n(s)$ and $b_n(s)$.

After these recollections we return to our Hamiltonians $nG_- \leqslant nK \leqslant nG_+$. Fix $q_1 \in V_n(q_0)$. By (17), the set

$$\mathcal{S}(n) := \big(\mathcal{S}(nG_-, q_0, q_1) \cup \mathcal{S}(nK, q_0, q_1)\big) \cap (n, n+1)$$

is finite. In particular,

$$\delta_n := \min\{s \in \mathcal{S}(nK, q_0, q_1) \mid s > n\} > 0.$$

Choose

$$a_n \in \big((n, n+1) \cap (n, n+\delta_n)\big) \setminus \mathcal{S}(n). \tag{23}$$

Then a_n is not in the action spectrum of nG_-, nK, and by Proposition 3·2,

$$x \in \mathcal{P}(nK, q_0, q_1) \cap D(\Sigma) \iff \mathcal{A}_{nK}(x) < a_n. \tag{24}$$

Our next goal is to show that $\mathrm{HF}_*^{a_n}(nG_-)$ and $\mathrm{HF}_*^{a_n/\sigma}(nG_+)$ are naturally isomorphic. This is a special case of a generalization of Lemma 4·3 stated as [**43**, proposition 1·1]. We give an ad hoc construction in the case at hand.

Choose $b_n \in (a_n, n+1)$ such that

$$[a_n, b_n] \cap \mathcal{S}(nG_-) = \varnothing, \tag{25}$$

and for $s \in [0, 1]$ define

$$a_n(s) = \frac{a_n}{1+\beta(s)(\sigma-1)}, \qquad b_n(s) = \frac{b_n}{1+\beta(s)(\sigma-1)}.$$

For (s, t) in the gray band bounded by the graphs of $a_n(s)$ and $b_n(s)$ we have $t \notin \mathcal{S}(nG_s)$ in view of the Non-crossing Lemma 3·3. Choose a partition $0 = s_0 < s_1 < \cdots < s_k < s_{k+1} = 1$ so fine that

$$b_n(s_{j+1}) > a_n(s_j), \qquad j = 0, \ldots, k. \tag{26}$$

Abbreviate $\mathfrak{a}_j = a_n(s_j)$ and $\mathfrak{G}_j = nG_{s_j}$. Then

$$\mathfrak{a}_j \notin \mathcal{S}(nG_s) \quad \text{for } s \in [s_j, s_{j+1}], \tag{27}$$

cf. Figure 3.

Together with Lemma 4·3 we find that

$$\Phi_{\mathfrak{G}_{j+1}\mathfrak{G}_j} : \mathrm{HF}_*^{\mathfrak{a}_j}(\mathfrak{G}_j) \longrightarrow \mathrm{HF}_*^{\mathfrak{a}_j}(\mathfrak{G}_{j+1})$$

is an isomorphism. Since $[\mathfrak{a}_{j+1}, \mathfrak{a}_j] \cap \mathcal{S}(\mathfrak{G}_{j+1}) = \emptyset$, we have $\mathrm{HF}_*^{\mathfrak{a}_j}(\mathfrak{G}_{j+1}) = \mathrm{HF}_*^{\mathfrak{a}_{j+1}}(\mathfrak{G}_{j+1})$, and so

$$\widehat{\Phi}_{\mathfrak{G}_{j+1}\mathfrak{G}_j} \equiv \Phi_{\mathfrak{G}_{j+1}\mathfrak{G}_j} \colon \mathrm{HF}_*^{\mathfrak{a}_j}(\mathfrak{G}_j) \longrightarrow \mathrm{HF}_*^{\mathfrak{a}_{j+1}}(\mathfrak{G}_{j+1})$$

is an isomorphism. Recalling $\mathfrak{a}_0 = a_n$ and $\mathfrak{a}_{k+1} = a_n/\sigma$ we obtain that the composition

$$\widehat{\Phi}_{nG_+nG_-} := \widehat{\Phi}_{\mathfrak{G}_{k+1}\mathfrak{G}_k} \circ \cdots \circ \widehat{\Phi}_{\mathfrak{G}_2\mathfrak{G}_1} \circ \widehat{\Phi}_{\mathfrak{G}_1\mathfrak{G}_0} \colon \mathrm{HF}_*^{a_n}(nG_-) \longrightarrow \mathrm{HF}_*^{a_n/\sigma}(nG_+)$$

is an isomorphism. Let

$$\mathrm{HF}_*(\iota)\colon \mathrm{HF}_*^{a_n/\sigma}(nG_+) \longrightarrow \mathrm{HF}_*^{a_n}(nG_+)$$

be the homomorphism induced by the inclusion $\mathrm{CF}_*^{a_n/\sigma}(nG_+) \to \mathrm{CF}_*^{a_n}(nG_+)$.

PROPOSITION 4·4. *For each k and n there is a commutative diagram of homomorphisms*

$$\begin{tikzcd}
 & \mathrm{HF}_k^{a_n/\sigma}(nG_+) \arrow[dr, "\mathrm{HF}_k(\iota)"] & \\
\mathrm{HF}_k^{a_n}(nG_-) \arrow[ur, "\widehat{\Phi}_{nG_+nG_-}"] \arrow[rr, "\Phi_{nG_+nG_-}"] \arrow[dr, "\Phi_{nKnG_-}"'] & & \mathrm{HF}_k^{a_n}(nG_+) \\
 & \mathrm{HF}_k^{a_n}(nK) \arrow[ur, "\Phi_{nG_+nK}"'] &
\end{tikzcd}$$

and $\widehat{\Phi}_{nG_+nG_-}$ *is an isomorphism.*

Proof. By construction, the isomorphism $\widehat{\Phi}_{nG_+nG_-}$ is induced by the composition of Floer chain maps

$$\widehat{\phi}_{\mathfrak{G}_{j+1}\mathfrak{G}_j} \colon \mathrm{CF}_*^{\mathfrak{a}_j}(\mathfrak{G}_j) \longrightarrow \mathrm{CF}_*^{\mathfrak{a}_{j+1}}(\mathfrak{G}_{j+1}) \subset \mathrm{CF}_*^{\mathfrak{a}_j}(\mathfrak{G}_{j+1}).$$

Therefore, $\mathrm{HF}_*(\iota) \circ \widehat{\Phi}_{nG_+nG_-}$ is induced by the composition of Floer chain maps

$$\phi_{\mathfrak{G}_{j+1}\mathfrak{G}_j} \colon \mathrm{CF}_*^{\mathfrak{a}_0}(\mathfrak{G}_j) \longrightarrow \mathrm{CF}_*^{\mathfrak{a}_0}(\mathfrak{G}_{j+1}).$$

By (22), this composition induces the same map in Floer homology as

$$\phi_{nG_+nG_-} \colon \mathrm{CF}_*^{\mathfrak{a}_0}(nG_-) \longrightarrow \mathrm{CF}_*^{\mathfrak{a}_0}(nG_+).$$

The upper triangle therefore commutes. The lower triangle commutes in view of $nG_- \leqslant nK \leqslant nG_+$ and according to (22).

COROLLARY 4·5. *We have that*

$$\dim \mathrm{HF}_k^{a_n}(nK, q_0, q_1; \mathbb{F}_p) \geqslant \operatorname{rank} \mathrm{HF}_k(\iota)\colon \mathrm{HF}_k^{a_n/\sigma}(nG_+) \to \mathrm{HF}_k^{a_n}(nG_+).$$

4·3. *From Floer homology to the homology of the based loop space*

In this section we use Corollary 4·5 and our assumption that M is energy hyperbolic to prove:

THEOREM 4·6. *Let (M, g) and K be as above. Then there exist $h > 0$, $p \in \mathbb{P}$ and $N \in \mathbb{N}$ depending only on (M, g) such that for all $n \geqslant N$ and for a_n as above the following holds. For all $q_0 \in M$ and all $q_1 \in V_n(q_0)$,*

$$\dim \mathrm{HF}_*^{a_n}(nK, q_0, q_1; \mathbb{F}_p) \geqslant e^{hn}.$$

Proof. For $q_0, q_1 \in M$ let $\Omega^1(M, q_0, q_1)$ be the space of all paths $q\colon [0,1] \to M$ of Sobolev class $W^{1,2}$ such that $q(0) = q_0$ and $q(1) = q_1$. Again, this space has a canonical Hilbert manifold structure. The energy functional $\mathcal{E}\colon \Omega^1(M, q_0, q_1) \to \mathbb{R}$ is defined as

$$\mathcal{E}(q) = \frac{1}{2}\int_0^1 |\dot{q}(t)|^2\, dt.$$

For $b > 0$ we consider the sublevel sets $\mathcal{E}^b(q_0, q_1) := \{q \in \Omega^1(M, q_0, q_1) \mid \mathcal{E}(q) \leqslant b\}$.

PROPOSITION 4·7. *For each k and n there is a commutative diagram of homomorphisms*

$$\begin{tikzcd} \mathrm{HF}_k^{a_n/\sigma}(nG_+, q_0, q_1; \mathbb{F}_p) \arrow[r, "\cong"] \arrow[d, "\mathrm{HF}_k(\iota)"'] & \mathrm{H}_k(\mathcal{E}^{na_n}(q_0, q_1); \mathbb{F}_p) \arrow[d, "\mathrm{H}_k(\iota)"] \\ \mathrm{HF}_k^{a_n}(nG_+, q_0, q_1; \mathbb{F}_p) \arrow[r, "\cong"] & \mathrm{H}_k(\mathcal{E}^{\sigma na_n}(q_0, q_1); \mathbb{F}_p) \end{tikzcd}$$

where the horizontal maps are isomorphisms and the right map $\mathrm{H}_k(\iota)$ *is induced by the inclusion* $\mathcal{E}^{na_n}(q_0, q_1) \hookrightarrow \mathcal{E}^{\sigma na_n}(q_0, q_1)$.

Proof. Let $L\colon TM \to \mathbb{R}$ be the Legendre transform of nG_+, let

$$\mathcal{E}_L(q) = \int_0^1 L(q(t), \dot{q}(t))dt$$

be the corresponding action functional on $\Omega^1(M, q_0, q_1)$, and let

$$\mathcal{E}_L^b(q_0, q_1) = \{q \in \Omega^1(M, q_0, q_1) \mid \mathcal{E}_L(q) \leqslant b\}.$$

Applying the Abbondandolo–Schwarz Theorem [**2**, theorem 3·1] to nG_+ and L, we obtain for each $b > 0$ an isomorphism

$$\mathrm{HF}_k^b(nG_+, q_0, q_1; \mathbb{F}_p) \xleftarrow{\Theta_k^b} \mathrm{HM}_k^b(L, q_0, q_1; \mathbb{F}_p).$$

Here, $\mathrm{HM}_k^b(L, q_0, q_1; \mathbb{F}_p)$ is the Morse homology "below level b" of $\mathcal{E}_L$ constructed in [**1**], see also [**2**, section 2]. The Abbondandolo–Schwarz chain isomorphisms

$$\mathrm{CF}_k^b(nG_+, q_0, q_1; \mathbb{F}_p) \xleftarrow{\theta_k^b} \mathrm{CM}_k^b(L, q_0, q_1; \mathbb{F}_p)$$

between the Morse and the Floer chain complexes commute with inclusions

$$\mathrm{CF}_k^b(nG_+, q_0, q_1; \mathbb{F}_p) \hookrightarrow \mathrm{CF}_k^{b'}(nG_+, q_0, q_1; \mathbb{F}_p) \quad \text{and}$$
$$\mathrm{CM}_k^b(L, q_0, q_1; \mathbb{F}_p) \hookrightarrow \mathrm{CM}_k^{b'}(L, q_0, q_1; \mathbb{F}_p)$$

for $b < b'$, see [**2**, p. 298]. Therefore, the induced diagram of homology groups commutes,

$$\begin{tikzcd} \mathrm{HF}_k^b(nG_+, q_0, q_1; \mathbb{F}_p) \arrow[d, "\mathrm{HF}_k(\iota)"'] & \mathrm{HM}_k^b(L, q_0, q_1; \mathbb{F}_p) \arrow[l, "\Theta_k^b"'] \arrow[d, "\mathrm{HM}_k(\iota)"] \\ \mathrm{HF}_k^{b'}(nG_+, q_0, q_1; \mathbb{F}_p) & \mathrm{HM}_k^{b'}(L, q_0, q_1; \mathbb{F}_p). \arrow[l, "\Theta_k^{b'}"'] \end{tikzcd}$$

Moreover, Abbondandolo and Majer constructed chain isomorphisms

$$\mathrm{CM}_k^b(L, q_0, q_1; \mathbb{F}_p) \xrightarrow{\tau_k^b} \mathrm{C}_k(\mathcal{E}_L^b(q_0, q_1); \mathbb{F}_p)$$

between the singular chain complexes and the Morse chain complexes, which commute with inclusions

$$\mathrm{CM}_k^b(L, q_0, q_1; \mathbb{F}_p) \hookrightarrow \mathrm{CM}_k^{b'}(L, q_0, q_1; \mathbb{F}_p) \quad \text{and}$$
$$\mathrm{C}_k(\mathcal{E}_L^b(q_0, q_1); \mathbb{F}_p) \hookrightarrow \mathrm{C}_k(\mathcal{E}_L^{b'}(q_0, q_1); \mathbb{F}_p)$$

for $b < b'$, see [**1**] and [**2**, section 2·3]. Therefore, the induced diagram of homology groups commutes,

$$\begin{tikzcd}
\mathrm{HM}_k^b(L, q_0, q_1; \mathbb{F}_p) \arrow[r, "T_k^b"] \arrow[d, "\mathrm{HM}_k(\iota)"'] & \mathrm{H}_k\left(\mathcal{E}_L^b(q_0, q_1); \mathbb{F}_p\right) \arrow[d, "\mathrm{H}_k(\iota)"] \\
\mathrm{HM}_k^{b'}(L, q_0, q_1; \mathbb{F}_p) \arrow[r, "T_k^{b'}"] & \mathrm{H}_k\left(\mathcal{E}_L^{b'}(q_0, q_1); \mathbb{F}_p\right).
\end{tikzcd}$$

Notice now that $L(q, v) = (1/n\sigma)(1/2)\,|v|^2$, whence $\mathcal{E}_L^b(q_0, q_1) = \mathcal{E}^{n\sigma b}(q_0, q_1)$ for all $b > 0$. The proposition follows.

Consider now the commutative diagram

$$\begin{tikzcd}
\mathrm{H}_k\left(\mathcal{E}^{na_n}(q_0, q_1); \mathbb{F}_p\right) \arrow[d] \arrow[dr, "\iota_k"] & \\
\mathrm{H}_k\left(\mathcal{E}^{\sigma na_n}(q_0, q_1); \mathbb{F}_p\right) \arrow[r] & \mathrm{H}_k\left(\Omega^1(M, q_0, q_1); \mathbb{F}_p\right)
\end{tikzcd}$$

induced by the inclusions $\mathcal{E}^{na_n}(q_0, q_1) \subset \mathcal{E}^{\sigma na_n}(q_0, q_1) \subset \Omega^1(M, q_0, q_1)$.

LEMMA 4·8. *For each $c \in (0, 1)$ there exists $A > 0$ depending only on c and (M, g) such that*

$$\dim \iota_k\, \mathrm{H}_k(\mathcal{E}^a(q_0, q_1); \mathbb{F}_p) \geqslant \dim \iota_k\, \mathrm{H}_k(\mathcal{E}^{ca}(q_0); \mathbb{F}_p)$$

for all k and all $a \geqslant A$.

Proof. Let ρ be the diameter of (M, g). Choose a path $\mathfrak{p}\colon [0, 1] \to M$ from q_0 to q_1 of length $\leqslant \rho$. Parametrize $\mathfrak{p}$ proportional to arc-length. Then $\mathcal{E}(\mathfrak{p}) \leqslant (1/2)\rho^2$. Let $\tau := (1/2)(c + 1) \in (0, 1)$. For $\gamma \in \Omega^1(M, q_0)$ define $\gamma *_\tau \mathfrak{p} \in \Omega^1(M, q_0, q_1)$ by

$$(\gamma *_\tau \mathfrak{p})\,(t) = \begin{cases} \gamma\left(\dfrac{t}{\tau}\right), & 0 \leqslant t \leqslant \tau, \\[2ex] \mathfrak{p}\left(\dfrac{t-\tau}{1-\tau}\right), & \tau \leqslant t \leqslant 1. \end{cases}$$

The map $\mathfrak{P}\colon \Omega^1(M, q_0) \to \Omega^1(M, q_0, q_1)$, $\gamma \mapsto \gamma *_\tau \mathfrak{p}$, is a homotopy equivalence with homotopy inverse $\Omega^1(M, q_0, q_1) \to \Omega^1(M, q_0)$, $\delta \mapsto \delta *_\tau \mathfrak{p}^{-1}$. Notice that

$$\mathcal{E}\,(\gamma *_\tau \mathfrak{p}) = \frac{1}{\tau}\mathcal{E}(\gamma) + \frac{1}{1-\tau}\mathcal{E}(\mathfrak{p}) \qquad \text{for all } \gamma \in \Omega^1(M, q_0).$$

Therefore,

$$\mathfrak{P}\,(\mathcal{E}^{ca}(q_0)) \subset \mathcal{E}^{\frac{ca}{\tau} + \frac{\mathcal{E}(\mathfrak{p})}{1-\tau}}(q_0, q_1) \qquad \text{for all } a.$$

Since $\mathcal{E}(\mathfrak{p}) \leqslant (1/2)\rho^2$, there exists $A > 0$ depending only on c and ρ such that

$$\frac{ca}{\tau} + \frac{\mathcal{E}(\mathfrak{p})}{1-\tau} \leqslant a \qquad \text{for all } a \geqslant A.$$

Hence

$$\mathfrak{P}\left(\mathcal{E}^{ca}(q_0)\right) \subset \mathcal{E}^{a}(q_0, q_1) \qquad \text{for all } a \geqslant A.$$

Since $\mathfrak{P}$ is a homotopy equivalence, it follows that

$$\dim \iota_k \, \mathrm{H}_k\left(\mathcal{E}^{ca}(q_0); \mathbb{F}_p\right) \leqslant \dim \iota_k \, \mathrm{H}_k\left(\mathcal{E}^{a}(q_0, q_1); \mathbb{F}_p\right) \qquad \text{for all } a \geqslant A,$$

as claimed.

Since (M, g) is energy hyperbolic, $h := C(M, g) > 0$. By definition of $C(M, g)$ and by Lemma 4·8 there exist $p \in \mathbb{P}$ and $N_0 \in \mathbb{N}$ such that for all $m \geqslant N_0$,

$$\sum_{k \geqslant 0} \dim \iota_k \, \mathrm{H}_k\left(\mathcal{E}^{\frac{1}{2}m^2}(q_0, q_1); \mathbb{F}_p\right) \geqslant e^{\frac{1}{\sqrt{2}}hm}.$$

Recall that $a_n \geqslant n$. Therefore there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$,

$$\sum_{k \geqslant 0} \operatorname{rank} \iota_k \geqslant e^{hn}.$$

Together with Proposition 4·7 we find that

$$\sum_{k \geqslant 0} \operatorname{rank} \mathrm{HF}_k(\iota) \geqslant \sum_{k \geqslant 0} \operatorname{rank} \iota_k \geqslant e^{hn}.$$

This and Corollary 4·5 yield Theorem 4·6.

Remark. The essential point in the proof of Theorem 4·6 is that the sum $\sum_{k \geqslant 0} \operatorname{rank} \iota_k$ grows exponentially with n. This is so by our assumption that M is energy hyperbolic. In the special case that the fundamental group of M has exponential growth, $\operatorname{rank} \iota_0$ grows exponentially with n. In another special case where M is simply connected and hyperbolic, a theorem of Gromov [**21**] guarantees the existence of a constant $c(M, g)$ such that ι_k is surjective if $n \geqslant c(M, g)k$, whence $\sum_{k \geqslant 0} \operatorname{rank} \iota_k$ grows exponentially with n.

5. *Proof of Theorem 1*

5·1. *From the growth of Floer homology to volume growth of $D_{q_0}(\Sigma)$*

Theorem 4·6 implies that the Riemannian volume of the sequence of submanifolds $\varphi_K^n\left(D_{q_0}(\Sigma)\right)$ grows exponentially. Indeed, fix $q_0 \in M$, and let $p \in \mathbb{P}$ and $N \in \mathbb{N}$ be as in Theorem 4·6. Let $n \geqslant N$ and pick $q_1 \in V_n(q_0)$. By (24), the generators of $\mathrm{CF}_k^{a_n}(nK, q_0, q_1; \mathbb{F}_p)$ correspond to $\varphi_K^n(D_{q_0}(\Sigma)) \cap D_{q_1}(\Sigma)$. Therefore,

$$\begin{aligned} \#\left(\varphi_K^n(D_{q_0}(\Sigma)) \cap D_{q_1}(\Sigma)\right) &= \dim \mathrm{CF}_*^{a_n}(nK, q_0, q_1; \mathbb{F}_p) \\ &\geqslant \dim \mathrm{HF}_*^{a_n}(nK, q_0, q_1; \mathbb{F}_p) \\ &\geqslant e^{hn}. \end{aligned} \tag{28}$$

Let μ_g be the Riemannian measure on (M, g). Let g^* be the Riemannian metric on T^*M induced by g, and for a submanifold $\sigma \subset T^*M$ let $\mu_{g^*}(\sigma)$ be the measure of σ with respect to the Riemannian measure on σ given by the Riemannian metric on σ induced by g^*. Note that $\pi : T^*M \to M$ is a Riemannian submersion with respect to the Riemannian metrics g^* and g, and recall from Corollary 3·5 that $V_n(q_0)$ is an open subset of M of full measure. Therefore,

$$\mu_{g^*}\left(\varphi_K^n(D_{q_0}(\Sigma))\right) \geqslant e^{hn} \mu_g\left(V_n(q_0)\right) = \mu_g(M)\, e^{hn}. \tag{29}$$

5·2. *From volume growth of the sublevel to topological entropy at the level*

If φ is a C^∞-smooth map of a compact manifold P, a geometric way of defining the topological entropy of φ was found by Yomdin and Newhouse in their seminal works [**45**] and [**33**]: Fix a Riemannian metric g on P. For $j \in \{1, \ldots, \dim P\}$ denote by S_j the set of smooth compact (not necessarily closed) j-dimensional submanifolds of P. The *jth volume growth* of φ is defined as

$$v_j(\varphi) = \sup_{\sigma \in S_j} \liminf_{n\to\infty} \frac{1}{n} \log \mu_g \left(\varphi^n(\sigma)\right),$$

and the *volume growth* of φ is defined as

$$v(\varphi) = \max_{1\leqslant j\leqslant \dim P} v_j(\varphi). \tag{30}$$

They do not depend on the choice of the Riemannian metric g used in their definition. Yomdin proved in [**45**] that $h_{\text{top}}(\varphi) \geqslant v(\varphi)$, and Newhouse proved in [**33**] that $h_{\text{top}}(\varphi) \leqslant v(\varphi)$. Thus

$$h_{\text{top}}(\varphi) = v(\varphi). \tag{31}$$

With $P = D(\Sigma)$ and $\varphi = \varphi_K|_{D(\Sigma)}$ we have, by (29), that $v_d\big(\varphi_K|_{D(\Sigma)}\big) > 0$. Hence, by (30) and (31),

$$h_{\text{top}}(\varphi_K|_{D(\Sigma)}) > 0. \tag{32}$$

Recall that $\Sigma = \partial D(\Sigma)$. The topological entropy of a flow is defined as the topological entropy of its time 1 map. Theorem 1 follows from (32) and the following

PROPOSITION 5·1. $h_{\text{top}}(\varphi_K|_\Sigma) = h_{\text{top}}(\varphi_K|_{D(\Sigma)})$.

Proof. For $s \in [0, 1]$ define the diffeomorphism ψ_s of T^*M by $\psi_s(q, p) = (q, sp)$. Abbreviate $s\Sigma = \psi_s(\Sigma)$. Recall from (11) that $K = f \circ F$ on $D(\Sigma)$. Also recall from (5) that $f'(r) = 1$ at $r = 1$. For $(q, p) \in \Sigma$ we thus have

$$X_{f\circ F}(q, sp) = (f'(F(q, sp))s)\, d\psi_s(q, p) X_{f\circ F}(q, p) = \sigma(s)\, d\psi_s(q, p) X_{f\circ F}(q, p). \tag{33}$$

For the latter equality we recalled that $F(q, sp) = s^2 F(q, p) = s^2$ and abbreviated

$$\sigma(s) = f'(s^2)s.$$

If $s \leqslant \varepsilon$, then $\sigma(s) = 0$, whence $X_{f\circ F}|_{s\Sigma} = 0$ and $h_{\text{top}}(\varphi_{f\circ F}|_{s\Sigma}) = 0$. If $s > \varepsilon$, then $\sigma(s) > 0$, and the identity (33) shows that ψ_s conjugates the flows $\varphi^t_{f\circ F}|_\Sigma$ and $\varphi^{\frac{1}{\sigma(s)}t}_{f\circ F}|_{s\Sigma}$. Topological entropy is an invariant of conjugacy, and scales like $h_{\text{top}}(\phi^{ct}) = |c|\, h_{\text{top}}(\phi^t)$ for $c \in \mathbb{R}$. Therefore,

$$h_{\text{top}}\left(\varphi^t_{f\circ F}|_{s\Sigma}\right) = \sigma(s)\, h_{\text{top}}\left(\varphi^{\frac{1}{\sigma(s)}t}_{f\circ F}|_{s\Sigma}\right) = \sigma(s)\, h_{\text{top}}\left(\varphi^t_{f\circ F}|_\Sigma\right).$$

If $s^2 \geqslant \varepsilon^2$, then $f'(s^2) = 1$, whence $\sigma(s) = s \leqslant 1$. If $s^2 \in (\varepsilon^2, \varepsilon)$, then $f'(s^2) \leqslant 2$, whence $\sigma(s) < 2s < 2\sqrt{\varepsilon} < 1$. If follows that

$$\sup_{s\in[0,1]} h_{\text{top}}\left(\varphi_{f\circ F}|_{s\Sigma}\right) = h_{\text{top}}\left(\varphi_{f\circ F}|_\Sigma\right).$$

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Proposition 5·1 can now be easily obtained from the variational principle for entropy. For convenience, we appeal to the following result of Bowen, [**5**, corollary 18], applied with $X = D(\Sigma)$, $Y = [0,1]$, $\phi_t = \varphi_K^t|_{D(\Sigma)}$ and $\pi\colon D(\Sigma) \to [0,1]$, $\pi(x) = s$ if $x \in s\Sigma$.

Let X, Y be compact metric spaces and $\phi_t\colon X \to X$ a continuous flow. Suppose that $\pi\colon X \to Y$ is a continuous map such that $\pi \circ \phi_t = \pi$. Then

$$h_{\rm top}(\varphi) = \sup_{y \in Y} h_{\rm top}(\phi|_{\pi^{-1}(y)}).$$

6. *Proof of Corollary 1*

It suffices to prove Corollary 1 for the Hamiltonian function $K\colon T^*M \to \mathbb{R}$ defined in (7). Write $q_0 = q$, and fix $q_1 \in M$ and $n \in \mathbb{N}$. Recall from the proof of Proposition 5·1 that for $s \in [0,1]$ the flow $\varphi_K^t|_{s\Sigma}$ is conjugate by ψ_s to the flow $\varphi_K^{\sigma(s)t}|_\Sigma$, where $\sigma(s) = f'(s^2)s$. We can choose the function f such that, in addition to the properties (5), we have

$$\sigma'(s) = 2sf''(s^2)s + f'(s^2) > 0 \quad \text{for all } s > \varepsilon.$$

Then the orbits of $\varphi_K^t|_\Sigma$ starting from Σ_{q_0} at $t = 0$ and arriving on Σ_{q_1} at $t \leqslant n$ are in bijection to points in $\varphi_K^n(D_{q_0}(\Sigma)) \bigcap D_{q_1}(\Sigma)$.

Assume now that

$$q_1 \in V(q_0) := \bigcap_{n \geqslant 1} V_n(q_0).$$

By Corollary 3·5 the set $V(q_0)$ has full measure in M. Let h be the constant from Theorem 4·6, that depends only on (M,g). For $q_1 \in V(q_0)$ we have, by (28), that

$$\#\big(\varphi_K^n(D_{q_0}(\Sigma)) \bigcap D_{q_1}(\Sigma)\big) \geqslant e^{hn}$$

provided that $n \geqslant N$ is large enough. The first assertion of Corollary 1 follows.

Choose a sequence $\{q_{1,j}\} \subset V(q_0)$ such that $\lim_{j\to\infty} q_{1,j} = q_1$. Let $N \in \mathbb{N}$ as chosen in the beginning of section 5·1. By (28), for each $j \in \mathbb{N}$ the intersection $\varphi_K^N(D_{q_0}(\Sigma)) \bigcap D_{q_{1,j}}(\Sigma)$ is non-empty. We thus find a sequence of φ_K^t-orbits γ_j on Σ such that $\gamma_j(0) \in \Sigma_{q_0}$ and $\gamma_j(t_j) \in \Sigma_{q_{1,j}}$ for some $t_j \leqslant N$. By the Arzelà–Ascoli theorem we obtain a φ_K^t-orbit γ on Σ from Σ_{q_0} to Σ_{q_1} arriving at $t \leqslant N$.

Remark 6·1. Using Floer homologies $\mathrm{HF}_*^{[b_n,a_n]}$ with suitable action windows, the assertions of Corollary 1 can be improved: *If $\pi_1(M)$ has exponential growth, then the exponential lower bound for $\nu_n(q,q',H)$ holds for all pairs q,q'. Moreover, for all $q,q' \in M$ there are infinitely many flow lines from S_qM to $S_{q'}M$,* see [**44**].

7. *An example*

The following example was pointed out to us by Gabriel Paternain. Following [**30**], we consider the group $G = \mathbf{Sol}$, which is the semi-direct product of $\mathbb{R}^2$ with $\mathbb{R}$, with coordinates $q = (x,y,z)$ and multiplication

$$(x,y,z) \star (x',y',z') = (x + e^z x', y + e^{-z} y', z + z').$$

The map $(x, y, z) \mapsto z$ is the epimorphism **Sol** $\to \mathbb{R}$ whose kernel is the normal subgroup $\mathbb{R}^2$. The group **Sol** is isomorphic to the matrix group

$$\begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix}.$$

It admits cocompact lattices. Indeed, let $A \in \mathrm{SL}(2, \mathbb{Z})$ be such that there is $P \in \mathrm{GL}(2, \mathbb{R})$ with

$$PAP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$$

and $\lambda > 1$. The injective homomorphism

$$\mathbb{Z}^2 \ltimes_A \mathbb{Z} \hookrightarrow \mathbf{Sol}$$

given by $(m, n, l) \mapsto (P(m, n), (\log \lambda)\, l)$ defines a cocompact lattice Γ in **Sol**. The closed 3-manifold $M := \Gamma \setminus \mathbf{Sol}$ is a 2-torus bundle over the circle with hyperbolic gluing map A. Since the group Γ has exponential growth, M is energy hyperbolic.

If we denote by p_x, p_y and p_z the momenta that are canonically conjugate to x, y and z, respectively, then the functions

$$\begin{aligned} M_x &= e^z p_x, \\ M_y &= e^{-z} p_y, \\ M_z &= p_z, \end{aligned}$$

are left-invariant functions on $T^*\mathbf{Sol}$. The 1-form θ defined by $\theta_q = e^{-z} dx$ is also left-invariant. The Hamiltonian on **Sol** defined by

$$2H = e^{2z}(p_x + e^{-z})^2 + e^{-2z} p_y^2 + p_z^2 = (M_x + 1)^2 + M_y^2 + M_z^2 \tag{34}$$

is left-invariant and hence descends to M. Observe that $H(q, p) = (1/2)|p + \theta_q|^2$, where the norm is induced by the left-invariant Riemannian metric

$$ds^2 = e^{-2z} dx^2 + e^{2z} dy^2 + dz^2$$

on **Sol**. Note that $\left|\theta_q\right| = 1$ at each $q \in M$, and that for each $k > 0$ the energy level $\Sigma_k := H^{-1}(k)$ is a 2-sphere bundle over the graph of θ. In terms of the functions M_x, M_y, M_z we have $\Sigma_k = M \times S_k$, where $S_k = \left\{(M_x + 1)^2 + M_y^2 + M_z^2 = 2k\right\}$. Since S_k encloses the origin iff $k > 1/2$, the hypersurface Σ_k is fiberwise starshaped with respect to the origin iff $k > 1/2$. Denote by φ_H the Hamiltonian flow of H. By Theorem 1, $h_{\mathrm{top}}(\varphi_H|_{\Sigma_k}) > 0$ if $k > 1/2$. The following proposition shows that the assumption in Theorem 1 that Σ is fiberwise starshaped with respect to the origin can, in general, not be omitted.

PROPOSITION 7·1. *$h_{\mathrm{top}}(\varphi_H|_{\Sigma_k}) > 0$ if and only if $k > 1/2$.*

Proof. The proof uses basic notions and results from smooth ergodic theory, as exposed in [**26**]. The Hamiltonian vector field of H is given by

$$X_H = \begin{cases} \dot{x} = (M_x + 1)e^z, & \dot{M}_x = M_x M_z, \\ \dot{y} = M_y e^{-z}, & \dot{M}_y = -M_y M_z, \\ \dot{z} = M_z, & \dot{M}_z = M_y^2 - M_x(M_x + 1). \end{cases} \tag{35}$$

The equations on the right-hand side describe the Euler vector field associated to X_H.

Fix $k > 0$, and abbreviate $\varphi = \varphi_H|_{\Sigma_k}$. By the variational principle for entropy,

$$h_{\text{top}}(\varphi) = \sup_{\mu \in \mathcal{M}(\varphi)} h_\mu(\varphi)$$

where $\mathcal{M}(\varphi)$ is the set of φ-invariant Borel probability measures on Σ_k, and $h_\mu(\varphi)$ is the entropy of φ with respect to the measure μ. By the Margulis–Ruelle inequality, in turn,

$$h_\mu(\varphi) \leqslant \int_{\Sigma_k} \chi_+(q, p)\, d\mu(q, p), \tag{36}$$

where $\chi_+(q, p)$ is the sum of the positive Lyapunov exponents of $(q, p) \in \Sigma_k$. As explained in [**6**, section 3·2], the Lyapunov exponents of φ can be computed using the projection **Sol** $\xrightarrow{z} \mathbb{R}$, and it turns out that

$$\chi_+(q, p) = \lim_{T \to \infty} \frac{1}{T} \left| \int_0^T M_z(\varphi^t(q, p))\, dt \right|. \tag{37}$$

We claim that for $k \leqslant 1/2$ the right-hand side of (36) vanishes for every invariant measure μ. Indeed, if $k < 1/2$ equation (34) implies that $M_x < 0$. This and (35) imply that $M_z = \dot{M}_x/M_x = (d/dt)\ln(-M_x)$, which, by (37), implies that $\chi_+(q, p) = 0$ for every $(q, p) \in \Sigma_k$.

Consider now the case $k = 1/2$. By (34) we have that $M_x(q, p) \leqslant 0$ for every $(q, p) \in \Sigma_k$. By the previous argument, if $\sup_{t\in\mathbb{R}} M_x(\varphi^t(q, p)) \neq 0$ then $\chi_+(q, p) = 0$. Suppose that (q, p) satisfies $\sup_{t\in\mathbb{R}} M_x(\varphi^t(q, p)) = 0$. It follows from (35) that $M_x M_y$ is a first integral of H. Since $\sup_{t\in\mathbb{R}} M_x(\varphi^t(q, p)) = 0$, this integral must vanish. Hence $M_x(q, p) = 0$ or $M_y(q, p) = 0$. If $M_x(q, p) = 0$, equation (34) yields $M_y(q, p) = 0$. By (35), $M_y(\varphi^t(q, p)) = 0$ for all t. Inspection of (35) shows that an orbit of the Euler vector field in the circle given by the intersection of the plane $M_y = 0$ with the sphere $(M_x + 1)^2 + M_y^2 + M_z^2 = 1$ is either the singularity $M_x = M_y = M_z = 0$ or a regular orbit that converges to the origin. But, by the Poincaré Recurrence Theorem, μ-almost every point (q, p) is recurrent. Consequently only the points with $M_x = M_y = M_z = 0$ may contribute to the integral in (36). These points, by (37), do not have positive Lyapunov exponents. We conclude that $h_{\text{top}}(\varphi) = 0$ if $k \leqslant 1/2$.

Recall that Theorem 1 implies $h_{\text{top}}(\varphi) > 0$ for $k > 1/2$. We give a different proof for this example: For $k > 1/2$, the Euler vector field has the singularities $p_\pm$ given by $M_x = M_y = 0$ and $M_z = \pm\sqrt{2k-1}$. The measure $\mu_M = dx \wedge dy \wedge dz$ is smooth on $M \times \{p_+\}$ and φ-invariant. Moreover, it is easy to see from [**6**, section 3·2] that the positive Lyapunov exponents of points in $M \times \{p_+\}$ are given by the magnetic flow restricted to $M \times \{p_+\}$. Let μ be the induced measure on Σ_k given by $\mu(A) = \mu_M(A \bigcap (M \times \{p_+\}))$ for every Borel set A. It follows from the definition of entropy that $h_\mu(\varphi) = h_{\mu_M}(\varphi|_{M\times\{p_+\}})$. By the Variational Principle, Pesin's formula, and (37),

$$h_{\text{top}}(\varphi) \geqslant h_\mu(\varphi) = h_{\mu_M}(\varphi|_{M\times\{p_+\}}) = \int_{\Sigma_k} \chi_+(q, p)\, d\mu(q, p) = \sqrt{2k-1} > 0,$$

finishing the proof of the proposition.

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